

Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations

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*Dedicated to Barry Simon
on the occasion of his 60th birthday*

Abstract

The purpose of the paper is to review a variety of recent developments in the theory of positive solutions of general linear elliptic and parabolic equations of second-order on noncompact Riemannian manifolds, and to point out a number of their consequences.

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1 Introduction

Positivity properties of general linear second-order elliptic and parabolic equations have been extensively studied over the recent decades (see for example [47, 68] and the references therein). The purpose of the present paper is to review a variety of recent developments in the theory of positive solutions of such equations and to point out a number of their (sometimes unexpected) consequences. The attention is focused on generalizations of positivity properties which were studied by Barry Simon in the special case of Schrödinger operators. Still, the selection of topics in this survey is incomplete, and is according to the author's working experience and taste. The reference list is far from being complete and serves only this exposé.

The outline of the paper is as follows. In Section 2, we introduce some fundamental notions that will be studied throughout the paper. In particular, we bring up the notions of the generalized principal eigenvalue, criticality and subcriticality of elliptic operators, and the Martin boundary. Section 3 is devoted to different types of perturbations and their properties. In Section 4, we study the behavior of critical operators under indefinite perturbations. In sections 5 and 6 we discuss some relationships between criticality theory and the theory of nonnegative solutions of the corresponding parabolic equations. More precisely, in Section 5 we deal with the large time behavior of the heat kernel, while in Section 6 we discuss sufficient conditions for the nonuniqueness of the positive Cauchy problem, and study intrinsic ultracontractivity.

In Section 7, we study the asymptotic behavior at infinity of eigenfunctions of Schrödinger operators. The phenomenon known in the mathematical physics literature as ‘localization of binding’, and the properties of the shuttle operator are discussed in sections 8 and 9, respectively. The exact asymptotics of the positive minimal Green function, and the explicit Martin integral representation theorem for positive solutions of general \mathbb{Z}^d -periodic elliptic operators on \mathbb{R}^d are reviewed in Section 10. We devote Section 11 to some relationships between criticality theory and Liouville theorems. In particular, we reveal that an old open problem of B. Simon (Problem 9.1) is completely solved (see Theorem 11.2). In Section 12 we study polynomially growing solutions of \mathbb{Z}^d -periodic equations on \mathbb{R}^d . We conclude the paper in Section 13 with criticality theory for the p -Laplacian with a potential term.

2 Principal eigenvalue, minimal growth and classification

Consider a noncompact, connected, smooth Riemannian manifold X of dimension d . For any subdomain $\Omega \subseteq X$, we write $D \Subset \Omega$ if \overline{D} is a compact subset of Ω . The ball of radius $r > 0$ and center at x_0 is denoted by $B(x_0, r)$. Let $f, g \in C(\Omega)$, we use the notation $f \asymp g$ on $D \subseteq \Omega$ if there exists a positive constant C such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x) \quad \text{for all } x \in D.$$

By **1**, we denote the constant function taking at any point the value 1.

We associate to any subdomain $\Omega \subseteq X$ an *exhaustion* of Ω , i.e. a sequence of smooth, relatively compact domains $\{\Omega_j\}_{j=1}^\infty$ such that $\Omega_1 \neq \emptyset$, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $\cup_{j=1}^\infty \Omega_j = \Omega$. For every $j \geq 1$, we denote $\Omega_j^* = \Omega \setminus \overline{\Omega_j}$. We

say that a function $f \in C(\Omega)$ *vanishes at infinity of Ω* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x)| < \varepsilon$ for all $x \in \Omega_N^*$.

We associate to any such exhaustion $\{\Omega_j\}_{j=1}^\infty$ a sequence $\{\chi_j(x)\}_{j=1}^\infty$ of smooth cutoff functions in Ω such that $\chi_j(x) \equiv 1$ in Ω_j , $\chi_j(x) \equiv 0$ in $\Omega \setminus \Omega_{j+1}$, and $0 \leq \chi_j(x) \leq 1$ in Ω . Let $0 < \alpha \leq 1$. For $W \in C^\alpha(\Omega)$, we denote $W_j(x) = \chi_j(x)W(x)$ and $W_j^*(x) = W(x) - W_j(x)$.

We consider a linear, second-order, elliptic operator P defined in a subdomain $\Omega \subset X$. Here P is an operator with *real* Hölder continuous coefficients which in any coordinate system $(U; x_1, \dots, x_d)$ has the form

$$P(x, \partial_x) = - \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^d b_i(x) \partial_i + c(x), \quad (2.1)$$

where $\partial_i = \partial/\partial x_i$. We assume that for each $x \in \Omega$ the real quadratic form $\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j$ is positive definite on \mathbb{R}^d .

We denote the cone of all positive (classical) solutions of the elliptic equation $Pu = 0$ in Ω by $\mathcal{C}_P(\Omega)$. We fix a reference point $x_0 \in \Omega_1$. From time to time, we consider the convex set

$$\mathcal{K}_P(\Omega) := \{u \in \mathcal{C}_P(\Omega) \mid u(x_0) = 1\}$$

of all *normalized* positive solutions. In case that the coefficients of P are smooth enough, we denote by P^* the formal adjoint of P .

Definition 2.1. For a (real valued) function $V \in C^\alpha(\Omega)$, let

$$\lambda_0(P, \Omega, V) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda V}(\Omega) \neq \emptyset\}$$

be the *generalized principal eigenvalue* of the operator P with respect to the (indefinite) weight V in Ω . We also denote

$$\lambda_\infty(P, \Omega, V) := \sup_{K \Subset \Omega} \lambda_0(P, \Omega \setminus K, V).$$

For a fixed P and Ω , and $V = \mathbf{1}$, we simply write $\lambda_0 := \lambda_0(P, \Omega, \mathbf{1})$ and $\lambda_\infty := \lambda_\infty(P, \Omega, \mathbf{1})$.

Definition 2.2. Let P be an elliptic operator of the form (2.1) which is defined on a smooth domain $D \Subset X$. we say that the *generalized maximum principle* for the operator P holds in D if for any $u \in C^2(D) \cap C(\overline{D})$, the inequalities $Pu \geq 0$ in D and $u \geq 0$ on ∂D imply that $u \geq 0$ in D .

It is well known that $\lambda_0(P, \Omega, \mathbf{1}) \geq 0$ if and only if the generalized maximum principle for the operator P holds true in any smooth subdomain $D \Subset \Omega$.

The following theorem is known as the *Allegretto-Piepenbrink theory*, it relates λ_0 and λ_∞ , in the symmetric case, with fundamental spectral quantities (see for example [1, 17, 76] and the references therein).

Theorem 2.3. *Suppose that P is symmetric on $C_0^\infty(\Omega)$, and that $\lambda_0 > -\infty$. Then λ_0 (resp. λ_∞) equals to the infimum of the spectrum (resp. essential spectrum) of the Friedrich's extension of P .*

Therefore, in the selfadjoint case, λ_0 can be characterized via the classical Rayleigh-Ritz variational formula. In the general case, a variational principle for λ_0 is given by the Donsker-Varadhan variational formula (which is a generalization of the Rayleigh-Ritz formula) and by some other variational formulas (see for example [51, 68]).

Definition 2.4. Let P be an elliptic operator defined in a domain $\Omega \subseteq X$. A function u is said to be a *positive solution of the operator P of minimal growth in a neighborhood of infinity in Ω* if $u \in \mathcal{C}_P(\Omega_j^*)$ for some $j \geq 1$, and for any $l > j$, and $v \in C(\Omega_l^* \cup \partial\Omega_l) \cap \mathcal{C}_P(\Omega_l^*)$, if $u \leq v$ on $\partial\Omega_l$, then $u \leq v$ on Ω_l^* .

Theorem 2.5 ([1]). *Suppose that $\mathcal{C}_P(\Omega) \neq \emptyset$. Then for any $x_0 \in \Omega$ the equation $Pu = 0$ has (up to a multiple constant) a unique positive solution v in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in Ω .*

By the well known theorem on the removability of isolated singularity [29], we have:

Definition 2.6. Suppose that $\mathcal{C}_P(\Omega) \neq \emptyset$. If the solution v of Theorem 2.5 has a nonremovable singularity at x_0 , then P is said to be a *subcritical operator* in Ω . If v can be (uniquely) continued to a positive solution \tilde{v} of the equation $Pu = 0$ in Ω , then P is said to be a *critical operator* in Ω , and the positive global solution \tilde{v} is called a *ground state* of the equation $Pu = 0$ in Ω . The operator P is said to be *supercritical* in Ω if $\mathcal{C}_P(\Omega) = \emptyset$.

Remarks 2.7. 1. In [74], B. Simon coined the terms ‘(sub)-(super)-critical operators’ for Schrödinger operators with short-range potentials which are defined on \mathbb{R}^d , where $d \geq 3$. The definition given in [74] is in terms of the exact (and particular) large time behavior of the heat kernel of such operators (see [75, p. 71] for the root of this terminology). In [43], M. Murata

generalized the above classification for Schrödinger operators which are defined in any subdomain of \mathbb{R}^d , $d \geq 1$. The definition of subcriticality given here is due to [52].

2. The notions of minimal growth and ground state were introduced by S. Agmon in [1].

3. For modified and stronger notions of subcriticality see [24, 52].

Outline of the proof of Theorem 2.5. Assume that $\mathcal{C}_P(\Omega) \neq \emptyset$ and fix $x_0 \in \Omega$. Then for every $j \geq 1$, the *Dirichlet Green function* $G_P^{\Omega_j}(x, y)$ for the operator P exists in Ω_j . It is the integral kernel such that for any $f \in C_0^\infty(\Omega)$, the function $u_j(x) := \int_{\Omega_j} G_P^{\Omega_j}(x, y)f(y) dy$ solves the Dirichlet boundary value problem

$$Pu = f \quad \text{in } \Omega_j, \quad u = 0 \quad \text{on } \partial\Omega_j.$$

It follows that $G_P^{\Omega_j}(\cdot, x_0) \in \mathcal{C}_P(\Omega_j \setminus \{x_0\})$. By the generalized maximum principle, $\{G_P^{\Omega_j}(x, x_0)\}_{j=1}^\infty$ is an increasing sequence which, by the Harnack inequality, converges uniformly in any compact subdomain of $\Omega \setminus \{x_0\}$ either to $G_P^\Omega(x, x_0)$, the positive *minimal Green function* of P in Ω with a pole at x_0 (and in this case P is subcritical in Ω) or to infinity.

In the latter case, fix $x_1 \in \Omega$, such that $x_1 \neq x_0$. It follows that the sequence $G_P^{\Omega_j}(\cdot, x_0)/G_P^{\Omega_j}(x_1, x_0)$ converges uniformly in any compact subdomain of $\Omega \setminus \{x_0\}$ to a ground state of the equation $Pu = 0$ in Ω , and in this case P is critical in Ω . \square

Corollary 2.8. (i) If P is subcritical in Ω , then for each $y \in \Omega$ the Green function $G_P^\Omega(\cdot, y)$ with a pole at y exists, and is a positive solution of the equation $Pu = 0$ of minimal growth in a neighborhood of infinity in Ω . Moreover, P is subcritical in Ω if and only if the equation $Pu = 0$ in Ω admits a positive supersolution which is not a solution.

(ii) The operator P is critical in Ω if and only if the equation $Pu = 0$ in Ω admits (up to a multiplicative constant) a unique positive supersolution. In particular, $\dim \mathcal{C}_P(\Omega) = 1$.

(iii) Suppose that P is symmetric on $C_0^\infty(\Omega)$ with respect to a smooth positive density V , and let \tilde{P} be the (Dirichlet) selfadjoint realization of P on $L^2(\Omega, V(x)dx)$. Assume that $\lambda \in \sigma_{\text{point}}(\tilde{P})$ admits a nonnegative eigenfunction φ , then $\lambda = \lambda_0$ and $P - \lambda_0 V$ is critical in Ω (see for example [43]).

(iv) The operator P is critical (resp. subcritical) in Ω if and only if P^* is critical (resp. subcritical) in Ω .

As was mentioned, (sub)criticality is related to the large time behavior of the heat kernel. Indeed, (sub)criticality can be also defined in terms

of the corresponding parabolic equation. Suppose that $\lambda_0 \geq 0$. For every $j \geq 1$, consider the Dirichlet heat kernel $k_P^{\Omega_j}(x, y, t)$ of the parabolic operator $L := \partial_t + P$ on $\Omega_j \times (0, \infty)$. So, for any $f \in C_0^\infty(\Omega)$, the function $u_j(x, t) = \int_{\Omega_j} k_P^{\Omega_j}(x, y, t) f(y) dy$ solves the initial-Dirichlet boundary value problem

$$Lu = 0 \text{ in } \Omega_j \times (0, \infty), \quad u = 0 \text{ on } \partial\Omega_j \times (0, \infty), \quad u = f \text{ on } \Omega_j \times \{0\}.$$

By the (parabolic) generalized maximum principle, $\{k_P^{\Omega_j}(x, y, t)\}_{j=1}^\infty$ is an increasing sequence which converges to $k_P^\Omega(x, y, t)$, the *minimal heat kernel* of the parabolic operator L in Ω .

Lemma 2.9. *Suppose that $\lambda_0 \geq 0$. Let $x, y \in \Omega$, $x \neq y$. Then*

$$\int_0^\infty k_P^\Omega(x, y, t) dt < \infty \quad (\text{resp. } \int_0^\infty k_P^\Omega(x, y, t) dt = \infty),$$

if and only if P is a subcritical (resp. critical) operator in Ω . Moreover, if P is subcritical operator in Ω , then

$$G_P^\Omega(x, y) = \int_0^\infty k_P^\Omega(x, y, t) dt. \quad (2.2)$$

For the proof of Lemma 2.9 see for example [68]. Note that if $\lambda < \lambda_0$, then the operator $P - \lambda$ is subcritical in Ω , and that for $\lambda \leq \lambda_0$, the heat kernel $k_{P-\lambda}^\Omega(x, y, t)$ of the operator $P - \lambda$ is equal to $e^{\lambda t} k_P^\Omega(x, y, t)$.

Subcriticality (criticality) can be defined also through a probabilistic approach. If the zero-order coefficient c of the operator P is equal to zero in Ω , then P is called a *diffusion operator*. In this case, $P\mathbf{1} = 0$, and therefore, P is not supercritical in Ω . Moreover, for such an operator P , one can associate a diffusion process corresponding to a solution of the generalized martingale problem for P in Ω . This diffusion process is either *transient* or *recurrent* in Ω . It turns out that a diffusion operator P is subcritical in Ω if and only if the associated diffusion process is transient in Ω (for more details see [68]). A Riemannian manifold X is called *parabolic* (resp. *non-parabolic*) if the Brownian motion, the diffusion process with respect to the Laplace-Beltrami operator on X , is recurrent (resp. transient) [32].

Suppose now that P is of the form (2.1), and P is not supercritical in Ω . Let $\varphi \in \mathcal{C}_P(\Omega)$. Then the operator P^φ acting on functions u by

$$P^\varphi u := \frac{1}{\varphi} P(\varphi u)$$

is a diffusion operator, and

$$k_{P^\varphi}^M(x, y, t) = \frac{1}{\varphi(x)} k_P^M(x, y, t) \varphi(y).$$

Therefore, P is subcritical in Ω if and only if P^φ is transient in Ω .

We have the following general convexity results.

Theorem 2.10 ([54]). (i) Let $V \in C^\alpha(\Omega)$, $V \neq 0$ and set

$$S_+ = S_+(P, \Omega, V) = \{\lambda \in \mathbb{R} \mid P - \lambda V \text{ is subcritical in } \Omega\}, \quad (2.3)$$

$$S_0 = S_0(P, \Omega, V) = \{\lambda \in \mathbb{R} \mid P - \lambda V \text{ is critical in } \Omega\}. \quad (2.4)$$

Then $S := S_+ \cup S_0 \subseteq \mathbb{R}$ is a closed interval and $S_0 \subset \partial S$. Moreover, if $S \neq \emptyset$, then S is bounded if and only if V changes its sign in Ω .

(ii) Let $W, V \in C^\alpha(\Omega)$, then the function $\lambda_0(\mu) := \lambda_0(P - \mu W, \Omega, V)$ is a concave function on the interval $\{\mu \in \mathbb{R} \mid |\lambda_0(\mu)| < \infty\}$.

Proof. For $0 \leq s \leq 1$, and $V_0, V_1 \in C^\alpha(\Omega)$, let

$$P_s := P + sV_1 + (1-s)V_0.$$

Assume that u_j are positive supersolutions of the equations $P_j u \geq 0$ in Ω , where $j = 0, 1$. It can be verified that for $0 < s < 1$, the function

$$u_s(x) := [u_0(x)]^{1-s} [u_1(x)]^s$$

is a positive supersolution of the equation $P_s u = 0$ in Ω . Moreover, for any $0 < s < 1$, $u_s \in \mathcal{C}_{P_s}(\Omega)$ if and only if $V_0 = V_1$, and $u_0, u_1 \in \mathcal{C}_{P_0}(\Omega)$ are linearly dependent. The lemma follows easily from this observation. \square

Corollary 2.11 ([54] and [76]). Suppose that $P_s := P + sV_1 + (1-s)V_0$ is subcritical in Ω for $s = 0, 1$. Then for $0 \leq s \leq 1$ we have

$$G_{P_s}^\Omega(x, y) \leq [G_{P_0}^\Omega(x, y)]^{1-s} [G_{P_1}^\Omega(x, y)]^s. \quad (2.5)$$

Remark 2.12. The dependence of λ_0 on the higher order coefficients of P is more involved. In [12] it was proved that in the class of uniformly elliptic operators with bounded coefficients which are defined on a bounded domain in \mathbb{R}^d , λ_0 is locally Lipschitz continuous as a function of the first-order coefficients of the operator P . A. Ancona [7] proved that under some assumptions, λ_0 is Lipschitz continuous with respect to a metric $\text{dist}(P_1, P_2)$ measuring the distance between two elliptic operators P_1 and P_2 in a certain class. Ancona's metric depends on the difference between *all* the coefficients of the operators P_1 and P_2 .

If P is subcritical in Ω , then $\mathcal{C}_P(\Omega)$ is in general not a one-dimensional cone. Nevertheless, one can construct the *Martin compactification* Ω_P^M of Ω with respect to the operator P (with a base point x_0), and obtain an integral representation of any solution in $\mathcal{C}_P(\Omega)$. More precisely, the *Martin compactification* is the compactification of Ω such that the function

$$K_P^\Omega(x, y) := \frac{G_P^\Omega(x, y)}{G_P^\Omega(x_0, y)} \quad \text{on } \Omega \times \Omega \setminus \{(x_0, x_0)\}$$

has a continuous extension $K_P^\Omega(x, \eta)$ to $\Omega \times (\Omega_P^M \setminus \{x_0\})$, and such that the set of functions $\{K_P^\Omega(\cdot, \eta)\}_{\eta \in \Omega_P^M}$ separates the points of Ω_P^M . The boundary of Ω_P^M is denoted by $\partial_P^M \Omega$ and is called the *Martin boundary* of Ω with respect to the operator P . For each $\xi \in \partial_P^M \Omega$, the function $K_P^\Omega(\cdot, \xi)$ is called the *Martin function of the pair (P, Ω) with a pole at ξ* . Note that for $\xi \in \partial_P^M \Omega$, we have $K_P^\Omega(\cdot, \xi) \in \mathcal{K}_P(\Omega)$. The set $\partial_{m,P}^M \Omega$ of all $\xi \in \partial_P^M \Omega$ such that $K_P^\Omega(\cdot, \xi)$ is an extreme point of the convex set $\mathcal{K}_P(\Omega)$ is called the *minimal Martin boundary* (for more details see [43, 47, 68, 81, and the references therein]).

The Martin representation theorem asserts that for any $u \in \mathcal{K}_P(\Omega)$ there exists a unique probability measure μ on $\partial_P^M \Omega$ which is supported on $\partial_{m,P}^M \Omega$ such that

$$u(x) = \int_{\partial_P^M \Omega} K_P^\Omega(x, \xi) d\mu(\xi).$$

There has been a great deal of work on explicit description of the Martin compactification and representation in many concrete examples (see for example [41, 47, 50, 68, 81, and the references therein]).

We present below two elementary examples of Martin compactifications. In Section 10 we discuss a recent result on the Martin compactification of a general periodic operator on \mathbb{R}^d .

Example 2.13. Let Ω be a smooth bounded domain in \mathbb{R}^d , and assume that the coefficients of P are (up to the boundary) smooth. Then $\partial_P^M \Omega$ is homeomorphic to $\partial\Omega$, the euclidian boundary of Ω , and for any $y \in \partial\Omega$,

$$K_P^\Omega(x, y) := \frac{\partial_\nu G_P^\Omega(x, y)}{\partial_\nu G_P^\Omega(x_0, y)}, \quad (2.6)$$

where ∂_ν denotes the inner normal derivative with respect to the second variable. Note that $\partial_\nu G_P^\Omega(\cdot, y)$ is the Poisson kernel at $y \in \partial\Omega$.

Example 2.14. Consider the equation $H_\lambda u := (-\Delta + \lambda)u = 0$ in \mathbb{R}^d . Then $\mathcal{C}_{H_\lambda}(\mathbb{R}^d) \neq \emptyset$ if and only if $\lambda \geq 0$. It is well known that $H_0 = -\Delta$ is critical

on in \mathbb{R}^d if and only if $d \leq 2$. Moreover,

$$G_{H_\lambda}^{\mathbb{R}^d}(x, y) = \begin{cases} \frac{\Gamma(\nu)|x-y|^{2-d}}{4\pi^{d/2}} & \lambda = 0, \text{ and } d \geq 3, \\ (2\pi)^{-d/2} \left(\frac{\sqrt{\lambda}}{|x-y|} \right)^\nu K_\nu(\sqrt{\lambda}|x-y|) & \lambda > 0, \end{cases}$$

where $\nu = (d-2)/2$, and K_ν is the modified Bessel function of order ν .

Clearly,

$$\lim_{|y| \rightarrow \infty} \frac{G_{-\Delta}^{\mathbb{R}^d}(x, y)}{G_{-\Delta}^{\mathbb{R}^d}(0, y)} = 1.$$

Therefore, the Martin compactification of \mathbb{R}^d with respect to the Laplacian is the one-point compactification of \mathbb{R}^d , and we obtained the positive Liouville theorem: $\mathcal{K}_{-\Delta}(\mathbb{R}^d) = \{\mathbf{1}\}$.

Suppose now that $\lambda > 0$. Then for any $\xi \in S^{d-1}$,

$$\lim_{\substack{y \rightarrow \xi, \\ |y| \rightarrow \infty}} \frac{G_{H_\lambda}^{\mathbb{R}^d}(x, y)}{G_{H_\lambda}^{\mathbb{R}^d}(0, y)} = e^{\sqrt{\lambda}\xi \cdot x},$$

and therefore, the Martin boundary of \mathbb{R}^d with respect to H_λ is the sphere at infinity. Clearly, all Martin functions are minimal. Furthermore, $u \in \mathcal{C}_{H_\lambda}(\mathbb{R}^d)$ if and only if there exists a positive finite measure μ on S^{d-1} such that

$$u(x) = \int_{S^{d-1}} e^{\sqrt{\lambda}\xi \cdot x} d\mu(\xi).$$

Remark 2.15. We would like to point out that criticality theory and Martin boundary theory are also valid for the class of weak solutions of elliptic equations in divergence form as well as for the class of strong solutions of strongly elliptic equations with locally bounded coefficients. For the sake of clarity, we prefer to concentrate on the class of classical solutions.

3 Perturbations

An operator P is critical in Ω if and only if any positive supersolution of the equation $Pu = 0$ in Ω is a solution (Corollary 2.8). Therefore, if P is critical in Ω and $V \in C^\alpha(\Omega)$ is a nonzero, nonnegative function, then for any $\lambda > 0$ the operator $P + \lambda V$ is subcritical and $P - \lambda V$ is supercritical in Ω . On the other hand, it can be shown that subcriticality is a stable property in

the following sense: if P is subcritical in Ω and $V \in C^\alpha(\Omega)$ has a *compact support*, then there exists $\epsilon > 0$ such that $P - \lambda V$ is subcritical for all $|\lambda| < \epsilon$, and the Martin compactifications Ω_P^M and $\Omega_{P-\lambda V}^M$ are homeomorphic for all $|\lambda| < \epsilon$ (for a more general result see Theorem 3.6). Therefore, a perturbation by a compactly supported potential (at least with a definite sign) is well understood.

In this section, we introduce and study a few general notions of perturbations related to positive solutions of an operator P of the form (2.1) by a (real valued) potential V . In particular, we discuss the behavior of the generalized principal eigenvalue, (sub)criticality, the Green function, and the Martin boundary under such perturbations. Further aspects of perturbation theory will be discussed in the following sections.

One facet of this study is the equivalence (or comparability) of the corresponding Green functions.

Definition 3.1. Let P_j , $j = 1, 2$, be two subcritical operators in Ω . We say that the Green functions $G_{P_1}^\Omega$ and $G_{P_2}^\Omega$ are *equivalent* (resp. *semi-equivalent*) if $G_{P_1}^\Omega \asymp G_{P_2}^\Omega$ on $\Omega \times \Omega \setminus \{(x, x) \mid x \in \Omega\}$ (resp. $G_{P_1}^\Omega(\cdot, y_0) \asymp G_{P_2}^\Omega(\cdot, y_0)$ on $\Omega \setminus \{y_0\}$ for some fixed $y_0 \in \Omega$).

Lemma 3.2 ([52]). *Suppose that the Green functions $G_{P_1}^\Omega$ and $G_{P_2}^\Omega$ are equivalent. Then there exists a homeomorphism $\Phi : \partial_{m, P_1}^M \Omega \rightarrow \partial_{m, P_2}^M \Omega$ such that for each minimal point $\xi \in \partial_{m, P_1}^M \Omega$, we have $K_{P_1}^\Omega(\cdot, \xi) \asymp K_{P_2}^\Omega(\cdot, \Phi(\xi))$ on Ω . Moreover, the cones $\mathcal{C}_{P_1}(\Omega)$ and $\mathcal{C}_{P_2}(\Omega)$ are homeomorphic.*

Remarks 3.3. 1. It is not known whether the equivalence of $G_{P_1}^\Omega$ and $G_{P_2}^\Omega$ implies that the cones $\mathcal{C}_{P_1}(\Omega)$ and $\mathcal{C}_{P_2}(\Omega)$ are *affine* homeomorphic.

2. Many papers deal with sufficient conditions, in terms of proximity near infinity in Ω between two given subcritical operators P_1 and P_2 , which imply that $G_{P_1}^\Omega$ and $G_{P_2}^\Omega$ are equivalent, or even that the cones $\mathcal{C}_{P_1}(\Omega)$ and $\mathcal{C}_{P_2}(\Omega)$ are affine homeomorphic, see Theorem 3.6 and [4, 7, 43, 46, 52, 53, 72, and the references therein].

We use the notation

$$E_+ = E_+(V, P, \Omega) := \{ \lambda \in \mathbb{R} \mid G_{P-\lambda V}^\Omega \text{ and } G_P^\Omega \text{ are equivalent} \},$$

$$sE_+ = sE_+(V, P, \Omega) := \{ \lambda \in \mathbb{R} \mid G_{P-\lambda V}^\Omega \text{ and } G_P^\Omega \text{ are semi-equivalent} \}.$$

The following notion was introduced in [53] and is closely related to the stability of $\mathcal{C}_P(\Omega)$ under perturbation by a potential V .

Definition 3.4. Let P be a subcritical operator in Ω , and let $V \in C^\alpha(\Omega)$. We say that V is a *small perturbation* of P in Ω if

$$\lim_{j \rightarrow \infty} \left\{ \sup_{x, y \in \Omega_j^*} \int_{\Omega_j^*} \frac{G_P^\Omega(x, z) |V(z)| G_P^\Omega(z, y)}{G_P^\Omega(x, y)} dz \right\} = 0. \quad (3.1)$$

The following notions of perturbations were introduced by M. Murata [46].

Definition 3.5. Let P be a subcritical operator in Ω , and let $V \in C^\alpha(\Omega)$.

(i) We say that V is a *semismall perturbation* of P in Ω if

$$\lim_{j \rightarrow \infty} \left\{ \sup_{y \in \Omega_j^*} \int_{\Omega_j^*} \frac{G_P^\Omega(x_0, z) |V(z)| G_P^\Omega(z, y)}{G_P^\Omega(x_0, y)} dz \right\} = 0. \quad (3.2)$$

(ii) We say that V is a *G-bounded perturbation* (resp. *G-semibounded perturbation*) of P in Ω if there exists a positive constant C such that

$$\int_{\Omega} \frac{G_P^\Omega(x, z) |V(z)| G_P^\Omega(z, y)}{G_P^\Omega(x, y)} dz \leq C \quad (3.3)$$

for all $x, y \in \Omega$ (resp. for some fixed $x \in \Omega$ and all $y \in \Omega \setminus \{x\}$).

(iii) We say that V is an *H-bounded perturbation* (resp. *H-semibounded perturbation*) of P in Ω if there exists a positive constant C such that

$$\int_{\Omega} \frac{G_P^\Omega(x, z) |V(z)| u(z)}{u(x)} dz \leq C \quad (3.4)$$

for all $x \in \Omega$ (resp. for some fixed $x \in \Omega$) and all $u \in \mathcal{C}_P(\Omega)$.

(iv) We say that V is an *H-integrable perturbation* of P in Ω if

$$\int_{\Omega} G_P^\Omega(x, z) |V(z)| u(z) dz < \infty \quad (3.5)$$

for all $x \in \Omega$ and all $u \in \mathcal{C}_P(\Omega)$.

Theorem 3.6 ([46, 53, 54]). *Suppose that P is subcritical in Ω . Assume that V is a small (resp. semismall) perturbation of P^* in Ω . Then $E_+ = S_+$ (resp. $sE_+ = S_+$), and $\partial S = S_0$. In particular, S_+ is an open interval.*

Suppose that V is a semismall perturbation of P^ in Ω , and $\lambda \in S_0$. Let φ_0 be the corresponding ground state. Then $\varphi_0 \asymp G_P^\Omega(\cdot, x_0)$ in Ω_1^* .*

Suppose that V is a semismall perturbation of P^ in Ω , and $\lambda \in S_+$. Then the mapping*

$$\Psi(u) := u(x) + \lambda \int_{\Omega} G_{P-\lambda V}^\Omega(x, z) V(z) u(z) dz \quad (3.6)$$

is an affine homeomorphism of $\mathcal{C}_P(\Omega)$ onto $\mathcal{C}_{P-\lambda V}(\Omega)$, which induces a homeomorphism between the corresponding Martin boundaries. Moreover, in the small perturbation case, we have $\Psi(u) \asymp u$ in Ω for all $u \in \mathcal{C}_P(\Omega)$.

Remarks 3.7. 1. Small perturbations are semismall [46], G -(resp. H -) bounded perturbations are G - (resp. H -) semibounded, and H -semibounded perturbations are H -integrable. On the other hand, if V is H -integrable and $\dim \mathcal{C}_P(\Omega) < \infty$, then V is H -semibounded [46, 52].

There are potentials which are H -semibounded perturbations but are neither H -bounded nor G -semibounded. We do not know of any example of a semismall (resp. G -semibounded) perturbation which is not a small (resp. G -bounded) perturbation. We are also not aware of any example of a H -bounded (resp. H -integrable) perturbation which is not G -bounded (resp. H -semibounded) [61].

2. Any small (resp. semismall) perturbation is G -bounded (resp. G -semibounded), and any G -(resp. semi) bounded perturbation is H -(resp. semi) bounded perturbation.

3. If V is a G -bounded (resp. G -semibounded) perturbation of P (resp. P^*) in Ω , then G_P^Ω and $G_{P-\lambda V}^\Omega$ are equivalent (resp. semi-equivalent) provided that $|\lambda|$ is small enough [46, 52, 53]. On the other hand, if G_P^Ω and G_{P+V}^Ω are equivalent (resp. semi-equivalent) and V has a definite sign, then V is a G -bounded (resp. G -semibounded) perturbation of P (resp. P^*) in Ω . In this case, by (2.5), the set E_+ (resp. sE_+) is an open half line which is contained in S_+ [54, Corollary 3.6]. There are sign-definite G -bounded (resp. G -semibounded) perturbations such that $E_+ \subsetneq S_+$ (resp. $sE_+ \subsetneq S_+$) [61, Example 8.6], [47, Theorem 6.5].

Note that, if V is a G -(resp. semi-) bounded perturbation of P (resp. P^*) in Ω and $\Theta \in C^\alpha(\Omega)$ is any function which vanishes at infinity of Ω , then clearly the function $\Theta(x)V(x)$ is a (resp. semi-) small perturbation of the operator P (resp. P^*) in Ω .

4. Suppose that G_P^Ω and $G_{P-|V|}^\Omega$ are equivalent (resp. semi-equivalent). Using the resolvent equation it follows that the best equivalence (resp. semi-equivalence) constants of G_P^Ω and $G_{P\pm|V_j^*|}^\Omega$ tend to 1 as $j \rightarrow \infty$ if and only if V is a (resp. semi-) small perturbation of P (resp. P^*) in Ω . Therefore, zero-order perturbations of the type studied by A. Ancona in [7] provide us with a huge and almost optimal class of examples of small perturbations. (see also [4, 43, 46, 53, and the references therein]).

A. Grigor'yan and W. Hansen [33] have introduced the following notions of perturbations.

Definition 3.8. Let P be a subcritical operator in Ω , and fix $h \in \mathcal{C}_P(\Omega)$. A nonnegative function V is called *h -big on Ω* if any solution v of the equation $(P + V)v = 0$ in Ω satisfying $0 \leq v \leq h$ is identically zero. V is *non- h -big on Ω* if V is not h -big on Ω .

Remark 3.9. If V is H -integrable perturbation of P , then it is non- h -big for any $h \in \mathcal{C}_P(\Omega)$ (see Proposition 11.1).

The following notion of perturbation does not involve Green functions.

Definition 3.10. Let P be a subcritical operator in $\Omega \subseteq X$. A function $V \in C^\alpha(\Omega)$ is said to be a *weak perturbation* of the operator P in Ω if the following condition holds true.

(*) For every $\lambda \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that the operator $P - \lambda V_n^*(x)$ is subcritical in Ω for any $n \geq N$.

A function $V \in C^\alpha(\Omega)$ is said to be a *weak perturbation* of a critical operator P in Ω if there exists a nonzero, nonnegative function $W \in C_0^\alpha(\Omega)$ such that the function V is a weak perturbation of the subcritical operator $P + W$ in Ω .

Remarks 3.11. 1. If V is a weak perturbation of P in Ω , then $\partial S = S_0$ and $\lambda_\infty(P, \Omega, \pm V) = \infty$ ([60], see also Theorem 7.1).

2. If V is a semismall perturbation of P in Ω , then $|V|$ is a weak perturbation of P in Ω , but G -bounded perturbations are not necessarily weak.

3. Let $d \geq 3$. By the Cwikel-Lieb-Rozenblum estimate, if $V \in L^{d/2}(\mathbb{R}^d)$, then $|V|$ is a weak perturbation of $-\Delta$ in \mathbb{R}^d . On the other hand, $(1 + |x|)^{-2}$ is not a weak perturbation of $-\Delta$ in \mathbb{R}^d , while for any $\varepsilon > 0$ the function $(1 + |x|)^{-(2+\varepsilon)}$ is a small perturbation of $-\Delta$ in \mathbb{R}^d , $d \geq 3$ [43, 52].

4 Indefinite weight

Consider the Schrödinger operator $H_\lambda := -\Delta - \lambda W$ in \mathbb{R}^d , where $\lambda \in \mathbb{R}$ is a spectral parameter and $W \in C_0^\infty(\mathbb{R}^d)$, $W \not\equiv 0$. Since $-\Delta$ is subcritical in \mathbb{R}^d if and only if $d \geq 3$, it follows that for $d \geq 3$ the Schrödinger operator H_λ has no bound states provided that $|\lambda|$ is sufficiently small. On the other hand, for $d = 1, 2$, B. Simon proved the following sharp result.

Theorem 4.1 ([73]). *Suppose that $d = 1, 2$, and let $W \in C_0^\infty(\mathbb{R}^d)$, $W \not\equiv 0$. Then $H_\lambda = -\Delta - \lambda W$ has a negative eigenvalue for all negative λ if and only if $\int_{\mathbb{R}^d} W(x) dx \leq 0$.*

The following result extends Theorem 4.1 to the case of a weak perturbation of a general critical operator in Ω .

Theorem 4.2 ([60]). *Let P be a critical operator in Ω , and $W \in C^\alpha(\Omega)$ a weak perturbation of the operator P in Ω . Denote by φ_0 (resp. φ_0^*) the ground state of the operator P (resp. P^*) in Ω such that $\varphi_0(x_0) = 1$ (resp. $\varphi_0^*(x_0) = 1$). Assume that $W\varphi_0\varphi_0^* \in L^1(\Omega)$.*

(i) *If there exists $\lambda < 0$ such that $P - \lambda W(x)$ is subcritical in Ω , then*

$$\int_{\Omega} W(x)\varphi_0(x)\varphi_0^*(x) dx > 0. \quad (4.1)$$

(ii) *Assume that for some nonnegative, nonzero function $V \in C_0^\alpha(\Omega)$ there exists $\tilde{\lambda} < 0$ and a positive constant C such that*

$$G_{P+V-\lambda W}^\Omega(x, x_0) \leq C\varphi_0(x) \quad \text{and} \quad G_{P+V-\lambda W}^\Omega(x_0, x) \leq C\varphi_0^*(x) \quad (4.2)$$

for all $x \in \Omega \setminus \Omega_1$ and $\tilde{\lambda} \leq \lambda < 0$. If the integral condition (4.1) holds true, then there exists $\lambda < 0$ such that $P - \lambda W(x)$ is subcritical in Ω .

(iii) *Suppose that W is a semismall perturbation of the operators $P + V$ and $P^* + V$ in Ω , where $V \geq 0$, $V \in C_0^\alpha(\Omega)$. Then there exists $\lambda < 0$ such that $P - \lambda W(x)$ is subcritical in Ω if and only if (4.1) holds true.*

5 Large time behavior of the heat kernel

As was already mentioned in Section 2, the large time behavior of the heat kernel is closely related to criticality (see for example Lemma 2.9). In the present section we elaborate this relation further more.

Suppose that $\lambda_0(P, \Omega, \mathbf{1}) \geq 0$. We consider the parabolic operator L

$$Lu = u_t + Pu \quad \text{on } \Omega \times (0, \infty). \quad (5.1)$$

We denote by $\mathcal{H}_P(\Omega \times (a, b))$ the cone of all nonnegative solutions of the equation $Lu = 0$ in $\Omega \times (a, b)$. Let $k_P^\Omega(x, y, t)$ be the heat kernel of the parabolic operator L in Ω .

If P is critical in Ω , we denote by φ_0 the ground state of P in Ω satisfying $\varphi_0(x_0) = 1$. The corresponding ground state of P^* is denoted by φ_0^* .

Definition 5.1. A critical operator P is said to be *positive-critical* in Ω if $\varphi_0\varphi_0^* \in L^1(\Omega)$, and *null-critical* in Ω if $\varphi_0\varphi_0^* \notin L^1(\Omega)$.

Theorem 5.2 ([55, 62]). *Suppose that $\lambda_0 \geq 0$. Then for each $x, y \in \Omega$*

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} k_P^\Omega(x, y, t) = \begin{cases} \frac{\varphi_0(x) \varphi_0^*(y)}{\int_\Omega \varphi_0(z) \varphi_0^*(z) dz} & \text{if } P - \lambda_0 \text{ is positive-critical,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have the following Abelian-Tauberian type relation

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} k_P^\Omega(x, y, t) = \lim_{\lambda \nearrow \lambda_0} (\lambda_0 - \lambda) G_{P-\lambda}^\Omega(x, y). \quad (5.2)$$

Remark 5.3. The first part of Theorem 5.2 has been proved by I. Chavel and L. Karp [13] in the *selfadjoint* case. Later, B. Simon gave a shorter proof for the selfadjoint case using the spectral theorem and elliptic regularity [77].

We next ask how fast $\lim_{t \rightarrow \infty} e^{\lambda_0 t} k_P^\Omega(x, y, t)$ is approached. It is natural to conjecture that the limit is approached equally fast for different points $x, y \in \Omega$. Note that in the context of Markov chains, such an (*individual*) *strong ratio limit property* is in general not true [14]. The following conjecture was raised by E. B. Davies [20] in the selfadjoint case.

Conjecture 5.4. *Let $Lu = u_t + P(x, \partial_x)u$ be a parabolic operator which is defined on $\Omega \subseteq X$. Fix a reference point $x_0 \in \Omega$. Then*

$$\lim_{t \rightarrow \infty} \frac{k_P^\Omega(x, y, t)}{k_P^\Omega(x_0, x_0, t)} = a(x, y) \quad (5.3)$$

exists and is positive for all $x, y \in \Omega$.

If Conjecture 5.4 holds true, then for any fixed $y \in \Omega$ the limit function $a(\cdot, y)$ is a positive solution of the equation $(P - \lambda_0)u = 0$ which is (up to a multiplicative function) a parabolic Martin function in $\mathcal{H}_P(\Omega \times \mathbb{R}_-)$ associated with any Martin sequence of the form (y, t_n) where $t_n \rightarrow -\infty$ (see [20, 63, and the references therein] for further partial results).

6 Nonuniqueness of the positive Cauchy problem and intrinsic ultracontractivity

In this section we discuss the uniqueness the Cauchy problem

$$\begin{cases} Lu := u_t + Pu = 0 & \text{on } \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \quad (6.1)$$

in the class of nonnegative continuous solutions. So, we always assume that $u_0 \in C(X)$, and $u_0 \geq 0$.

Definition 6.1. A solution of the positive Cauchy problem in $\Omega_T := \Omega \times [0, T)$ with initial data u_0 is a nonnegative continuous function in Ω_T satisfying $u(x, 0) = u_0(x)$, and $Lu = 0$ in $\Omega \times (0, T)$ in the classical sense.

We say that the *uniqueness of the positive Cauchy problem* (UP) for the operator L in Ω_T holds, when any two solutions of the positive Cauchy problem satisfying the same initial condition are identically equal in Ω_T .

Let $u \in \mathcal{C}_P(\Omega)$. By the parabolic generalized maximum principle, either

$$\int_{\Omega} k(x, y, t) u(y) dy = u(x) \text{ for some (and hence for all) } x \in \Omega, t > 0, \quad (6.2)$$

or

$$\int_{\Omega} k(x, y, t) u(y) dy < u(x) \text{ for some (and hence for all) } x \in \Omega, t > 0, \quad (6.3)$$

see for example [19]. Note that both sides of (6.3) are solutions of the positive Cauchy problem (6.1) with the same initial data $u_0 = u$. Therefore, in order to show that UP does not hold for the operator L in Ω , it is sufficient to show that (6.3) holds true for some $u \in \mathcal{C}_P(\Omega)$. It is easy to show [19] that (6.3) holds true if and only if there exists $\lambda < 0$ such that

$$-\lambda \int_{\Omega} G_{P-\lambda}^{\Omega}(x, y) u(y) dy < u(x) \quad (6.4)$$

for some (and hence for all) $x \in \Omega$. Furthermore, it follows from [45] that (6.4) is satisfied if

$$\int_{\Omega} G_P^{\Omega}(x, y) u(y) dy < \infty \quad (6.5)$$

for some (and hence for all) $x \in \Omega$. Thus, we have:

Corollary 6.2. *If $\mathbf{1}$ is an H -integrable perturbation of a subcritical operator P in Ω , then the positive Cauchy problem is not uniquely solvable.*

Remarks 6.3. 1. A positive solution $u \in \mathcal{C}_P(\Omega)$ which satisfies (6.2) is called a *positive invariant solution*. If $P\mathbf{1} = 0$ and (6.2) holds for $u = \mathbf{1}$ one says that L *conserves probability in Ω* (see [32]). We note that if P is critical, then the ground state φ_0 is a positive invariant solution. It turns out that there exists a complete Riemannian manifold X which does not admit any positive invariant harmonic function, while $\lambda_0(-\Delta, X, \mathbf{1}) = 0$ [57].

2. For necessary and sufficient conditions for UP, see [36, 48] and the references therein.

The following important notion was introduced by E. B. Davies and B. Simon for Schrödinger operators [21, 22, 23].

Definition 6.4. Suppose that P is symmetric. The Schrödinger semigroup e^{-tP} associated with the heat kernel $k_P^\Omega(x, y, t)$ is called *intrinsic ultracontractive* (IU) if $P - \lambda_0$ is positive-critical in Ω with a ground state φ_0 , and for each $t > 0$ there exists a positive constant C_t such that

$$C_t^{-1}\varphi_0(x)\varphi_0(y) \leq k_P^\Omega(x, y, t) \leq C_t\varphi_0(x)\varphi_0(y) \quad \forall x, y \in \Omega.$$

Remarks 6.5. 1. If e^{-tP} is IU, then

$$\lim_{t \rightarrow \infty} e^{\lambda_0 t} k_P^\Omega(x, y, t) = \frac{\varphi_0(x)\varphi_0(y)}{\int_\Omega [\varphi_0(z)]^2 dz} \quad (6.6)$$

uniformly in $\Omega \times \Omega$ (see for example [8], cf. Theorem 5.2).

2. If Ω is a bounded uniformly Hölder domain of order $0 < \alpha < 2$, then $e^{-t(-\Delta)}$ is IU on Ω [8].

3. Let $\alpha \geq 0$. Then $e^{-t(-\Delta + |x|^\alpha)}$ is IU on \mathbb{R}^d if and only if $\alpha > 2$.

Intrinsic ultracontractivity is closely related to perturbation theory of positive solutions and hence to UP, as the following recent result of M. Murata and M. Tomisaki demonstrates.

Theorem 6.6 ([46, 49]). *Suppose that P is a subcritical symmetric operator, and that the Schrödinger semigroup e^{-tP} is IU on Ω . Then $\mathbf{1}$ is a small perturbation of P on Ω . In particular, UP does not hold in Ω .*

On the other hand, there are planar domains such that $\mathbf{1}$ is a small perturbation of the Laplacian, but the semigroup $e^{-t(-\Delta)}$ is not IU (see [9] and [61]).

7 Asymptotic behavior of eigenfunctions

In this section, we assume that P is symmetric and discuss relationships between perturbation theory, Martin boundary, and the asymptotic behavior of weighted eigenfunctions in some general cases (for other relationships between positivity and decay of Schrödinger eigenfunctions see, [2, 76, 78]).

Theorem 7.1. (i) *Let $V \in C^\alpha(\Omega)$ be a positive function. Suppose that P is a symmetric, nonnegative operator on $L^2(\Omega, V(x)dx)$ with a domain $C_0^\infty(\Omega)$. Assume that V is a weak perturbation of the operator P in Ω . Suppose that*

P admits a (Dirichlet) selfadjoint realization \tilde{P} on $L^2(\Omega, V(x)dx)$. Then \tilde{P} has a purely discrete nonnegative spectrum (that is, $\sigma_{\text{ess}}(\tilde{P}) = \emptyset$). Moreover,

$$\sigma(\tilde{P}) = \sigma_{\text{discrete}}(\tilde{P}) = \sigma_{\text{point}}(\tilde{P}) = \{\lambda_n\}_{n=0}^{\infty},$$

where $\lim_{n \rightarrow \infty} \lambda_n = \infty$. In particular, if $\lambda_0 := \lambda_0(P, \Omega, V) > 0$, then the natural embedding $E : \mathcal{H} \rightarrow L^2(\Omega, V(x)dx)$ is compact, where \mathcal{H} is the completion of $C_0^\infty(\Omega)$ with respect to the inner product induced by the corresponding quadratic form.

(ii) Assume further that P is subcritical and V is a semismall perturbation of the operator P in Ω . Let $\{\varphi_n\}_{n=0}^{\infty}$ be the set of the corresponding eigenfunctions ($P\varphi_n = \lambda_n V\varphi_n$). Then for every $n \geq 1$ there exists a positive constant C_n such that

$$|\varphi_n(x)| \leq C_n \varphi_0(x). \quad (7.1)$$

(iii) For every $n \geq 1$, the function φ_n/φ_0 has a continuous extension ψ_n up to the Martin boundary $\partial_P^M \Omega$, and ψ_n satisfies

$$\psi_n(\xi) = (\psi_0(\xi))^{-1} \lambda_n \int_{\Omega} K_P^\Omega(z, \xi) V(z) \varphi_n(z) dz = \frac{\lambda_n \int_{\Omega} K_P^\Omega(z, \xi) V(z) \varphi_n(z) dz}{\lambda_0 \int_{\Omega} K_P^\Omega(z, \xi) V(z) \varphi_0(z) dz}$$

for every $\xi \in \partial_P^M \Omega$, where ψ_0 is the continuous extension of $\varphi_0/G_P^\Omega(\cdot, x_0)$ to the Martin boundary $\partial_P^M \Omega$.

Remarks 7.2. 1. By [21], the semigroup $e^{-t\tilde{P}}$ is IU if and only if the pointwise eigenfunction estimate (7.1) holds true with $C_n = c_t \exp(t\lambda_n) \|\varphi_n\|_2$, for every $t > 0$ and $n \geq 1$. Here c_t is a positive function of t which may be taken as the function such that $k_P^\Omega(x, y, t) \leq c_t \varphi_0(x) \varphi_0(y)$, where k_P^Ω is the corresponding heat kernel. It follows that if $e^{-t\tilde{P}}$ is IU, then the pointwise eigenfunction estimate (7.1) holds true with $C_n = \inf_{t>0} \{c_t \exp(t\lambda_n)\} \|\varphi_n\|_2$. We note that in general $\{C_n\}$ is unbounded [30].

Recall that if $e^{-t\tilde{P}}$ is IU, then $\mathbf{1}$ is a small perturbation of P (see Theorem 6.6). In particular, part (iii) of Theorem 7.1 implies that if $e^{-t\tilde{P}}$ is IU, then for any $n \geq 1$, the quotient φ_n/φ_0 has a continuous extension ψ_n up to the Martin boundary $\partial_P^M \Omega$.

2. M. Murata [44] proved part (ii) of Theorem 7.1 for the special case of bounded Lipschitz domains. See also [35] for related results on the asymptotic behavior of eigenfunctions of Schrödinger operators in \mathbb{R}^d .

8 Localization of binding

Let $V \in C^\alpha(\mathbb{R}^d)$ and $R \in \mathbb{R}^d$, throughout this section we use the notation $V^R(x) := V(x - R)$. For $j = 1, 2$, let V_j be small perturbations of the Laplacian in \mathbb{R}^d , $d \geq 3$, and assume that the operators $P_j := -\Delta + V_j(x)$ are nonnegative on $C_0^\infty(\Omega)$. We consider the Schrödinger operator

$$P_R := -\Delta + V_1(x) + V_2^R(x) \quad (8.1)$$

defined on \mathbb{R}^d , and its ground state energy $E(R) := \lambda_0(P_R, \mathbb{R}^d, \mathbf{1})$. In this section we discuss the asymptotic behavior of $E(R)$ as $|R| \rightarrow \infty$, a problem which was studied by M. Klaus and B. Simon in [38, 74] (see also [56, 68]). The motivation for studying the asymptotic behavior of $E(R)$ comes from a remarkable phenomenon known as the Efimov effect for a three-body Schrödinger operator (for more details, see for example [80]).

Definition 8.1. *Let $d \geq 3$. The space of functions*

$$K_d^\infty := \left\{ V \in C^\alpha(\mathbb{R}^d) \mid \lim_{M \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{|z| > M} \frac{|V(z)|}{|x - z|^{d-2}} dz = 0 \right\} \quad (8.2)$$

is called the Kato class at infinity.

Remark 8.2. Let $d \geq 3$. If $V \in K_d^\infty$, then V is a small perturbation of the Laplacian in \mathbb{R}^d .

Theorem 8.3 ([56]). *Let $d \geq 3$. For $j = 1, 2$, let $V_j(x) \in K_d^\infty$ be two functions such that the operators $P_j = -\Delta + V_j(x)$ are subcritical in \mathbb{R}^d . Then there exists $r_0 > 0$ such that the operator P_R is subcritical for any $R \in \mathbb{R}^d \setminus B(0, r_0)$. In particular, $E(R) = 0$ for all $|R| \geq r_0$.*

Assume now that the operators $P_j = -\Delta + V_j(x)$, $j = 1, 2$, are critical in \mathbb{R}^d . It turns out that in this case, there exists $r_0 > 0$ such that $E(R) < 0$ for $|R| \geq r_0$, but the asymptotic behavior of $E(R)$ depends on the dimension d , as the following theorems demonstrate (cf. [38, the remarks in pp. 84 and 87]).

Theorem 8.4 ([80]). *Let $d = 3$. Assume that the potentials V_j , $j = 1, 2$ satisfy $|V_j(x)| \leq C\langle x \rangle^{-\beta}$ on \mathbb{R}^3 , where $\langle x \rangle := (1 + |x|^2)^{1/2}$, $\beta > 2$, and $C > 0$. Suppose that $P_j = -\Delta + V_j(x)$ is critical in \mathbb{R}^3 for $j = 1, 2$.*

Then there exists $r_0 > 0$ such that the operator P_R is supercritical for any $R \in \mathbb{R}^3 \setminus B(0, r_0)$. Moreover, $E(R)$ satisfies

$$\lim_{|R| \rightarrow \infty} |R|^2 E(R) = -\beta^2 < -1/4, \quad (8.3)$$

where β is the unique root of the equation $s = e^{-s}$.

Theorem 8.5 ([58]). *Let $d = 4$. Assume that for $j = 1, 2$ the operators $P_j = -\Delta + V_j(x)$ are critical in \mathbb{R}^4 , where $V_j \in C_0^\alpha(\mathbb{R}^4)$.*

Then there exists $r_0 > 0$ such that the operator P_R is supercritical for any $R \in \mathbb{R}^4 \setminus B(0, r_0)$. Moreover, there exists a positive constant C such that $E(R)$ satisfies

$$-C|R|^{-2} \leq E(R) \leq -C^{-1}|R|^{-2}(\log |R|)^{-1} \quad \text{for all } |R| \geq r_0. \quad (8.4)$$

Theorem 8.6 ([56]). *Let $d \geq 5$. Suppose that V_j , $j = 1, 2$ satisfy $|V_j(x)| \leq C\langle x \rangle^{-\beta}$ in \mathbb{R}^d , where $\beta > d - 2$, and $C > 0$. Assume that the operators $P_j = -\Delta + V_j(x)$, $j = 1, 2$, are critical in \mathbb{R}^d .*

Then there exists $r_0 > 0$ such that the operator P_R is supercritical for any $R \in \mathbb{R}^d \setminus B(0, r_0)$. Moreover, there exists a positive constant C such that $E(R)$ satisfies

$$-C|R|^{2-d} \leq E(R) \leq -C^{-1}|R|^{2-d} \quad \text{for all } |R| \geq r_0. \quad (8.5)$$

What distinguishes $d \geq 5$ from $d = 3, 4$, is that for a short-range potential V , the ground state of a critical operator $-\Delta + V(x)$ in \mathbb{R}^d is in $L^2(\mathbb{R}^d)$ if and only if $d \geq 5$ (see [75] and Theorem 3.6).

9 The shuttle operator

In this section we present an intrinsic criterion which distinguishes between subcriticality, criticality and supercriticality of the operator P in Ω . This criterion depends only on the norm of a certain linear operator S , called the *shuttle operator* which is defined on $C(\partial D)$, where $D \Subset \Omega$.

The shuttle operator was introduced for Schrödinger operators on \mathbb{R}^d in [15, 16, 83, 84]. Using Feynman-Kac-type formulas [79], F. Gesztesy and Z. Zhao [28, 84] have studied the shuttle operator for Schrödinger operators in \mathbb{R}^d with short-range potentials (see also [27]), and its relation to the following problem posed by B. Simon.

Problem 9.1 ([75, 76]). *Let $V \in L_{\text{loc}}^2(\mathbb{R}^2)$. Show that if the equation $(-\Delta + V)u = 0$ on \mathbb{R}^2 admits a positive L^∞ -solution, then $-\Delta + V$ is critical.*

Gesztesy and Zhao used the shuttle operator and proved that for *short-range* potentials on \mathbb{R}^2 , the above condition is a necessary and sufficient condition for criticality (see also [42] and Theorem 3.6 for similar results, and Theorem 11.2 for the complete solution). On the other hand, Gesztesy and Zhao showed in [27, Example 4.6] that there is a critical Schrödinger

operator on \mathbb{R} with ‘almost’ short-range potential such that its ground state behaves logarithmically.

Let P be an elliptic operator of the form (2.1) which is defined on Ω . We assume that the following assumption **(A)** holds:

- (A)** There exist four smooth, relatively compact subdomains Ω_j , $0 \leq j \leq 3$, such that $\overline{\Omega_j} \subset \Omega_{j+1}$, $j = 0, 1, 2$, and such that $\mathcal{C}_P(\Omega_3) \neq \emptyset$ and $\mathcal{C}_P(\Omega_0^*) \neq \emptyset$.

Remarks 9.2. 1. If assumption **(A)** is not satisfied, then we shall say that the spectral radius of the shuttle operator is infinity. In this case, it is clear that P is supercritical in Ω .

2. Assumption **(A)** does not imply that $\mathcal{C}_P(\Omega) \neq \emptyset$.

Fix an exhaustion $\{\Omega_j\}_{j=0}^\infty$ of Ω , such that Ω_j satisfy assumption **(A)** for $0 \leq j \leq 3$. By assumption **(A)** the Dirichlet problem

$$Pu = 0 \quad \text{in } \Omega_2, \quad u = f \quad \text{on } \partial\Omega_2 \quad (9.1)$$

is uniquely solved in Ω_2 for any $f \in C(\partial\Omega_2)$, and we denote the corresponding operator from $C(\partial\Omega_2)$ into $C(\Omega_2)$ by T_{Ω_2} . Moreover, for every $f \in C(\partial\Omega_1)$, one can uniquely solve the exterior Dirichlet problem in the outer domain Ω_1^* , with ‘zero’ boundary condition at infinity of Ω . So, we have an operator $T_{\Omega_1^*} : C(\partial\Omega_1) \rightarrow C(\Omega_1^*)$ defined by

$$T_{\Omega_1^*} f(x) := \lim_{j \rightarrow \infty} u_{f,j}(x),$$

where $u_{f,j}$ is the solution of the Dirichlet boundary value problem:

$$Pu = 0 \quad \text{in } \Omega_1^* \cap \Omega_j, \quad u = f \quad \text{on } \partial\Omega_1^*, \quad u = 0 \quad \text{on } \partial(\Omega_1^* \cap \Omega_j) \setminus \partial\Omega_1^*.$$

For any open set D and $F \Subset D$, we denote by R_F^D the restriction map $f \mapsto f|_F$ from $C(D)$ into $C(F)$. The *shuttle operator* $S : C(\partial\Omega_1) \rightarrow C(\partial\Omega_1)$ is defined as follows:

$$S := R_{\partial\Omega_1}^{\Omega_2} T_{\Omega_2} R_{\partial\Omega_2}^{\Omega_1^*} T_{\Omega_1^*}. \quad (9.2)$$

We denote the spectral radius of the operator S by $r(S)$. We have

Theorem 9.3 ([59]). *The operator P is subcritical, critical, or supercritical in Ω according to whether $r(S) < 1$, $r(S) = 1$, or $r(S) > 1$.*

The proof of Theorem 9.3 in [59] is purely analytic and relies on the observation that (in the nontrivial case) S is a positive compact operator defined on the Banach space $C(\partial\Omega_1)$. Therefore, the Krein-Rutman theorem implies that there exists a simple principal eigenvalue $\nu_0 > 0$, which is equal to the norm (and also to the spectral radius) of S , and that the corresponding principal eigenfunction is strictly positive. It turns out, that the generalized maximum principle holds in any smooth subdomain $D \Subset \Omega$ if and only if $\nu_0 \leq 1$, and that $\nu_0 < 1$ if and only if P admits a positive minimal Green function in Ω .

The shuttle operator can be used to prove localization of binding for certain *nonselfadjoint* critical operators (see [59]).

10 Periodic operators

In this section we restrict the form of the operator. Namely, we assume that P is defined on \mathbb{R}^d and that the coefficients of P are \mathbb{Z}^d -periodic. For such operators, we introduce a function Λ that plays a crucial role in our considerations. Its properties were studied in detail in [3, 37, 41, 50, 67]. Consider the function $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by the condition that the equation $Pu = \Lambda(\xi)u$ on \mathbb{R}^d has a positive *Bloch solution* of the form

$$u_\xi(x) = e^{\xi \cdot x} \varphi_\xi(x), \quad (10.1)$$

where $\xi \in \mathbb{R}^d$, and φ_ξ is a positive \mathbb{Z}^d -periodic function.

- Theorem 10.1.**
1. The value $\Lambda(\xi)$ is uniquely determined for any $\xi \in \mathbb{R}^d$.
 2. The function Λ is bounded from above, strictly concave, analytic, and has a nonzero gradient for any $\xi \in \mathbb{R}^d$ except at its maximum point.
 3. For $\xi \in \mathbb{R}^d$, consider the operator $P(\xi) := e^{-\xi \cdot x} P e^{\xi \cdot x}$ on the torus \mathbb{T}^d . Then $\Lambda(\xi)$ is the principal eigenvalue of $P(\xi)$ with a positive eigenfunction φ_ξ . Moreover, $\Lambda(\xi)$ is algebraically simple.
 4. The Hessian of $\Lambda(\xi)$ is nondegenerate at all points $\xi \in \mathbb{R}^d$.

Let us denote

$$\Lambda_0 = \max_{\xi \in \mathbb{R}^d} \Lambda(\xi). \quad (10.2)$$

It follows from [3, 41, 67] that $\Lambda_0 = \lambda_0$, and that $P - \Lambda_0$ is critical if and only if $d = 1, 2$ (see also Corollary 11.5). Thus, in the self-adjoint case, Λ_0

coincides with the bottom of the spectrum of the operator P . Assume that $\Lambda_0 \geq 0$. Then Theorem 10.1 implies that the zero level set

$$\Xi = \left\{ \xi \in \mathbb{R}^d \mid \Lambda(\xi) = 0 \right\} \quad (10.3)$$

is either a strictly convex compact analytic surface in \mathbb{R}^d of dimension $d - 1$ (this is the case if and only if $\Lambda_0 > 0$), or a singleton (this is the case if and only if $\Lambda_0 = 0$).

In a recent paper [50], M. Murata and T. Tsuchida have studied the exact asymptotic behavior at infinity of the positive minimal Green function and the Martin boundary of such periodic elliptic operators on \mathbb{R}^d .

Suppose that $\Lambda_0 = \Lambda(\xi_0) > 0$. Then P is subcritical, and for each s in the unit sphere S^{d-1} there exists a unique $\xi_s \in \Xi$ such that

$$\xi_s \cdot s = \sup_{\xi \in \Xi} \{\xi \cdot s\}.$$

For $s \in S^{d-1}$ take an orthonormal basis of \mathbb{R}^d of the form $\{e_{s,1}, \dots, e_{s,d-1}, s\}$. For $\xi \in \mathbb{R}^d$, let φ_ξ and φ_ξ^* be periodic positive solutions of the equation $P(\xi)u = \Lambda(\xi)u$ and $P^*(\xi)u = \Lambda(\xi)u$ on \mathbb{T}^d , respectively, such that

$$\int_{\mathbb{T}^d} \varphi_\xi(x) \varphi_\xi^*(x) dx = 1.$$

Theorem 10.2 ([50]). *1. Suppose that $\Lambda_0 > 0$. Then the minimal Green function $G_P^{\mathbb{R}^d}$ of P on \mathbb{R}^d has the following asymptotics as $|x - y| \rightarrow \infty$:*

$$G_P^{\mathbb{R}^d}(x, y) = \frac{|\nabla \Lambda(\xi_s)|^{(d-3)/2} e^{-(x-y) \cdot \xi_s} \varphi_{\xi_s}(x) \varphi_{\xi_s}^*(y)}{(2\pi|x-y|)^{(d-1)/2} [\det(-e_{s,j} \cdot \text{Hess} \Lambda(\xi_s) e_{s,k})]^{1/2}} [1 + O(|x-y|^{-1})],$$

where $s := (x - y)/|x - y|$.

2. Suppose that $\Lambda_0 = \Lambda(\xi_0) = 0$ and $d \geq 3$. Then the minimal Green function $G_P^{\mathbb{R}^d}$ of P on \mathbb{R}^d has the following asymptotics as $|x - y| \rightarrow \infty$:

$$G_P^{\mathbb{R}^d}(x, y) = \frac{2^{-1} \pi^{-d/2} \Gamma(\frac{d-2}{2}) e^{-(x-y) \cdot \xi_0} \varphi_{\xi_0}(x) \varphi_{\xi_0}^*(y)}{\{\det[\text{Hess} \Lambda(\xi_0)]\}^{1/2} [-\text{Hess} \Lambda(\xi_0)]^{-1/2} (x-y)^{d-2}} [1 + O(|x-y|^{-1})].$$

Combining the results in [3, 50], we have the following Martin representation theorem.

Theorem 10.3 ([3, 50]). *Let Ξ be the set of all $\xi \in \mathbb{R}^d$ such that the equation $Pu = 0$ admits a positive Bloch solution $u_\xi(x) = e^{\xi \cdot x} \varphi_\xi(x)$ with*

$\varphi_\xi(0) = 1$. Then u is a positive Bloch solution if and only if u is a minimal Martin function of the equation $Pu = 0$ in \mathbb{R}^d . Moreover, all Martin functions are minimal. Furthermore, $u \in \mathcal{C}_P(\mathbb{R}^d)$ if and only if there exists a positive finite measure μ on Ξ such that

$$u(x) = \int_{\Xi} u_\xi(x) d\mu(\xi).$$

Theorem 10.3 (except the result that all Martin functions are minimal) was extended by V. Lin and the author to a manifold with a group action [41]. It is assumed that X is a noncompact manifold equipped with an action of a group G such that $GV = X$ for a compact subset $V \subseteq X$, and that the operator P is a G -invariant operator on X of the form (2.1). If G is finitely generated, then the set of all normalized positive solutions of the equation $Pu = 0$ in X which are also eigenfunctions of the G -action is a real analytic submanifold Ξ in an appropriate finite-dimensional vector space \mathcal{H} . Moreover, if Ξ is not a singleton, then it is the boundary of a strictly convex body in \mathcal{H} . If the group G is *nilpotent*, then any positive solution in $\mathcal{C}_P(X)$ can be uniquely represented as an integral of solutions over Ξ . In particular, $u \in \mathcal{C}_P(X)$ is a positive minimal solution if and only if it is a positive solution which is also an eigenfunction of the G -action.

11 Liouville theorems for Schrödinger operators and Criticality

The existence and nonexistence of nontrivial bounded solutions of the equation $Pu = 0$ are closely related to criticality theory as the following results demonstrate (see also Section 12).

Proposition 11.1 ([31],[61, Lemma 3.4]). *Suppose that V is a nonzero, nonnegative function such that V is an H -integrable perturbation of a subcritical operator P in Ω and let $u \in \mathcal{C}_P(\Omega)$. Then for any $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{C}_{P+\varepsilon V}(\Omega)$ which satisfies $0 < u_\varepsilon \leq u$ and the resolvent equation*

$$u_\varepsilon(x) = u(x) - \varepsilon \int_{\Omega} G_{P+\varepsilon V}^\Omega(x, z) V(z) u(z) dz. \quad (11.1)$$

In particular, if $P\mathbf{1} = 0$, then for any $\varepsilon > 0$ the operator $P + \varepsilon V$ admits a nonzero bounded solution.

In [18, Theorem 5], D. Damanik, R. Killip, and B. Simon proved a result which, formulated in the following new way, reveals a complete answer to

Problem 9.1 posed by B. Simon in [75, 76] (see also [28, 42] and Theorem 3.6). An alternative proof based on criticality theory is presented below.

Theorem 11.2 ([18]). *Let $d = 1$ or 2 , and $q \in L^2_{\text{loc}}(\mathbb{R}^d)$. Suppose that $H_q := -\Delta + q$ has a bounded positive solution in $\mathcal{C}_{H_q}(\mathbb{R}^d)$. If $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and both $H_{q \pm V} \geq 0$, then $V = 0$. In other words, H_q is critical.*

Proof. Theorem 2.10 implies that we should indeed show that H_q is critical. Assume that H_q is subcritical. Take a nonzero nonnegative W with a compact support. Then by Theorem 3.6, there exists $\varepsilon > 0$ such that $H_{q-\varepsilon W} \geq 0$. Let $M < N$. For $d = 1$ take the cutoff function

$$a_{M,N}(x) := \begin{cases} 0 & |x| > N, \\ 1 & |x| \leq M, \\ 1 - \frac{|x|-M}{N-M} & M < |x| \leq N, \end{cases}$$

and for $d = 2$

$$a_{M,N}(x) := \begin{cases} 0 & |x| > N, \\ 1 & |x| \leq M, \\ \frac{\log N - \log |x|}{\log N - \log M} & M < |x| \leq N. \end{cases}$$

Let ψ be a positive bounded solution of the equation $H_q u = 0$ in \mathbb{R}^d . Then for appropriate N, M with $M, N \rightarrow \infty$ (see [18]), we have

$$0 < c < \varepsilon \int_{\mathbb{R}^d} W(a_{M,N}\psi)^2 dx \leq \int_{\mathbb{R}^d} [|\nabla(a_{M,N}\psi)|^2 + q(a_{M,N}\psi)^2] dx = \int_{\mathbb{R}^d} |\nabla a_{M,N}|^2 \psi^2 dx \rightarrow 0,$$

and this is a contradiction. \square

Remarks 11.3. 1. Theorem 11.2 is related to Theorem 1.7 in [11] which claims that for $d = 1, 2$, if H_q admits a bounded solution that changes its sign, then $\lambda_0 < 0$. This claim and Theorem 11.2 do not hold for $d \geq 3$ [10].

2. For other relationships between perturbation theory of positive solutions and Liouville theorem see [32, 33].

After submitting the first version of the present article to the editors, we proved the following result which generalized Theorem 11.2 and the Liouville type theorems in [11].

Theorem 11.4 ([64]). *Let $\Omega \subset X$ be a domain. Consider two Schrödinger operators defined on Ω of the form*

$$P_j := -\nabla \cdot (A_j \nabla) + V_j \quad j = 0, 1, \quad (11.2)$$

such that $V_j \in L^p_{\text{loc}}(\Omega; \mathbb{R})$ for some $p > d/2$, and $A_j : \Omega \rightarrow \mathbb{R}^{d^2}$ are measurable matrix valued functions such that for any $K \Subset \Omega$ there exists $\mu_K > 1$ such that

$$\mu_K^{-1} I_d \leq A_j(x) \leq \mu_K I_d \quad \forall x \in K, \quad (11.3)$$

where I_d is the d -dimensional identity matrix.

Assume that the following assumptions hold true.

- (i) *The operator P_1 is critical in Ω . Denote by $\varphi \in \mathcal{C}_{P_1}(\Omega)$ its ground state.*
- (ii) *$\lambda_0(P_0, \Omega, \mathbf{1}) \geq 0$, and there exists a real function $\psi \in H^1_{\text{loc}}(\Omega)$ such that $\psi_+ \neq 0$, and $P_0 \psi \leq 0$ in Ω , where $u_+(x) := \max\{0, u(x)\}$.*
- (iii) *The following matrix inequality holds*

$$\psi^2(x) A_0(x) \leq C \varphi^2(x) A_1(x) \quad \text{a. e. in } \Omega, \quad (11.4)$$

where $C > 0$ is a positive constant.

Then the operator P_0 is critical in Ω , and ψ is its ground state. In particular, $\dim \mathcal{C}_{P_0}(\Omega) = 1$ and $\lambda_0(P_0, \Omega, \mathbf{1}) = 0$.

The proof of Theorem 11.4 relies on Theorem 13.6.

Corollary 11.5 ([67]). *Assume that the coefficients of the elliptic operator $P := -\nabla \cdot (A \nabla) + V$ are \mathbb{Z}^d -periodic on \mathbb{R}^d . Then the operator $P - \lambda_0$ is critical in \mathbb{R}^d if and only if $d \leq 2$.*

Remark 11.6. One can use [41] to extend Corollary 11.5 to the case of equivariant Schrödinger operators on cocompact coverings. Let X be a noncompact nilpotent covering of a compact Riemannian manifold. Suppose that $P := -\Delta + V$ is an equivariant operator on X with respect to its nilpotent deck group G . Then $P - \lambda_0$ is critical in X if and only if G has a normal subgroup of finite index isomorphic to \mathbb{Z}^d for $d \leq 2$.

12 Polynomially growing solutions and Liouville Theorems

Let $H = -\Delta + V$ be a Schrödinger operator on \mathbb{R}^d . Then Šnol's theorem asserts that, under some assumptions on the potential V , if H admits a polynomially growing solution of the equation $Hu = 0$ in \mathbb{R}^d , then $0 \in \sigma(H)$. Šnol's theorem was generalized by many authors including B. Simon, see for example [17, 76] and [71].

In [39, 40] the structure of the space of all polynomially growing solutions of a periodic elliptic operator (or a system) of order m on an abelian cover of a compact Riemannian manifold was studied. An important particular case of the general results in [39, 40] is a real, second-order \mathbb{Z}^d -periodic elliptic operator P of the form (2.1) which is defined on \mathbb{R}^d . In this case, we can use the information about positive solutions of such equations described in Section 10 and the results of [39] to obtain the precise structure and dimension of the space of polynomially growing solutions.

Definition 12.1. 1. Let $N \geq 0$. We say that *the Liouville theorem of order N for the equation $Pu = 0$* holds true in \mathbb{R}^d , if the space $V_N(P)$ of solutions of the equation $Pu = 0$ in \mathbb{R}^d that satisfy $|u(x)| \leq C(1 + |x|)^N$ for all $x \in \mathbb{R}^d$ is of finite dimension.

2. The *Fermi surface* F_P of the operator P consists of all vectors $\zeta \in \mathbb{C}^d$ such that the equation $Pu = 0$ has a nonzero *Bloch solution* of the form $u(x) = e^{i\zeta \cdot x} p(x)$, where p is a \mathbb{Z}^d -periodic function.

For a general \mathbb{Z}^d -periodic elliptic operator P of any order, we have:

Theorem 12.2 ([39]). 1. *If the Liouville theorem of an order $N \geq 0$ for the equation $Pu = 0$ holds true, then it holds for any order.*

2. *The Liouville theorem holds true if and only if the number of points in the real Fermi surface $F_P \cap \mathbb{R}^d$ is finite.*

For second-order operators with real coefficients, we have:

Theorem 12.3 ([39]). *Let P be a \mathbb{Z}^d -periodic operator on \mathbb{R}^d of the form (2.1) such that $\Lambda_0 \geq 0$. Then*

1. *The Liouville theorem holds vacuously if $\Lambda(0) > 0$, i.e., the equation $Lu = 0$ does not admit any nontrivial polynomially growing solution.*
2. *If $\Lambda(0) = 0$ and $\Lambda_0 > 0$, then the Liouville theorem holds for P , and*

$$\dim V_N(P) = \binom{d + N - 1}{N}.$$

3. If $\Lambda(0) = 0$ and $\Lambda_0 = 0$, then the Liouville theorem holds for P , and

$$\dim V_N(P) = \binom{d+N}{N} - \binom{d+N-2}{N-2},$$

which is the dimension of the space of all harmonic polynomials of degree at most N in d variables.

4. Any solution $u \in V_N(P)$ of the equation $Pu = 0$ can be represented as

$$u(x) = \sum_{|j| \leq N} x^j p_j(x)$$

with \mathbb{Z}^d -periodic functions p_j .

13 Criticality theory for the p -Laplacian with potential term

Positivity properties of quasilinear elliptic equations defined on a domain $\Omega \subset \mathbb{R}^d$, and in particular, those with the p -Laplacian term in the principal part, have been extensively studied over the recent decades (see for example [5, 6, 25, 26, 34, 82] and the references therein).

Let $p \in (1, \infty)$, and let Ω be a general domains in \mathbb{R}^d . Denote by $\Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ the p -Laplacian operator, and let $V \in L_{\text{loc}}^\infty(\Omega)$ be a given (real) potential. Throughout this section we always assume that

$$Q(u) := \int_{\Omega} (|\nabla u|^p + V|u|^p) \, dx \geq 0 \quad \forall u \in C_0^\infty(\Omega), \quad (13.1)$$

that is, the functional Q is nonnegative on $C_0^\infty(\Omega)$. In [66], K. Tintarev and the author studied (sub)criticality properties for positive weak solutions of the corresponding Euler-Lagrange equation

$$\frac{1}{p} Q'(v) := -\Delta_p(v) + V|v|^{p-2}v = 0 \quad \text{in } \Omega, \quad (13.2)$$

along the lines of criticality theory for second-order linear elliptic operators that was discussed in sections 2–4.

Definition 13.1. We say that the functional Q is *subcritical* in Ω (or Q is *strictly positive* in Ω) if there is a strictly positive continuous function W in Ω such that

$$Q(u) \geq \int_{\Omega} W|u|^p \, dx \quad \forall u \in C_0^\infty(\Omega). \quad (13.3)$$

Definition 13.2. We say that a sequence $\{u_n\} \subset C_0^\infty(\Omega)$ is a *null sequence*, if $u_n \geq 0$ for all $n \in \mathbb{N}$, and there exists an open set $B \Subset \Omega$ such that $\int_B |u_n|^p dx = 1$, and

$$\lim_{n \rightarrow \infty} Q(u_n) = \lim_{n \rightarrow \infty} \int_{\Omega} (|\nabla u_n|^p + V|u_n|^p) dx = 0. \quad (13.4)$$

We say that a positive function $\varphi \in C_{\text{loc}}^1(\Omega)$ is a *ground state* of the functional Q in Ω if φ is an $L_{\text{loc}}^p(\Omega)$ limit of a null sequence. If $Q \geq 0$, and Q admits a ground state in Ω , we say that the functional Q is *critical* in Ω . The functional Q is *supercritical* in Ω if $Q \not\geq 0$ on $C_0^\infty(\Omega)$.

The following is a generalization of the Allegretto-Piepenbrink theorem.

Theorem 13.3 (see [66]). *Let Q be a functional of the form (13.1). Then the following assertions are equivalent*

- (i) *The functional Q is nonnegative on $C_0^\infty(\Omega)$.*
- (ii) *Equation (13.2) admits a global positive solution.*
- (iii) *Equation (13.2) admits a global positive supersolution.*

The definition of positive solutions of minimal growth in a neighborhood of infinity in Ω in the linear case (Definition 2.4) is naturally extended to solutions of the equation $Q'(u) = 0$.

Definition 13.4. A positive solution u of the equation $Q'(u) = 0$ in Ω_j^* is said to be a *positive solution of the equation $Q'(u) = 0$ of minimal growth in a neighborhood of infinity in Ω* if for any $v \in C(\Omega_l^* \cup \partial\Omega_l)$ with $l > j$, which is a positive solution of the equation $Q'(u) = 0$ in Ω_l^* , the inequality $u \leq v$ on $\partial\Omega_l$, implies that $u \leq v$ on Ω_l^* .

If $1 < p \leq d$, then for each $x_0 \in \Omega$, any positive solution v of the equation $Q'(u) = 0$ in a punctured neighborhood of x_0 has either a removable singularity at x_0 , or

$$v(x) \sim \begin{cases} |x - x_0|^{\alpha(d,p)} & p < d, \\ -\log|x - x_0| & p = d, \end{cases} \quad \text{as } x \rightarrow x_0, \quad (13.5)$$

where $\alpha(d, p) := (p-d)/(p-1)$, and $f \sim g$ means that $\lim_{x \rightarrow x_0} [f(x)/g(x)] = C$ for some $C > 0$ (see [29] for $p = 2$, and [69, 70, 82, 66] for $1 < p \leq d$). The following result is an extension to the p -Laplacian of Theorem 2.5.

Theorem 13.5 ([66]). *Suppose that $1 < p \leq d$, and Q is nonnegative on $C_0^\infty(\Omega)$. Then for any $x_0 \in \Omega$ the equation $Q'(u) = 0$ has (up to a multiple*

constant) a unique positive solution v in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in Ω . Moreover, v is either a global minimal solution of the equation $Q'(u) = 0$ in Ω , or v has a nonremovable singularity at x_0 .

The main result of this section is as follows.

Theorem 13.6 ([66]). *Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $V \in L_{\text{loc}}^\infty(\Omega)$, and $p \in (1, \infty)$. Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$. Then*

- (a) *The functional Q is either subcritical or critical in Ω .*
- (b) *If the functional Q admits a ground state v , then v satisfies (13.2).*
- (c) *The functional Q is critical in Ω if and only if (13.2) admits a unique positive supersolution.*
- (d) *Suppose that $1 < p \leq d$. Then the functional Q is critical (resp. subcritical) in Ω if and only if there is a unique (up to a multiplicative constant) positive solution φ_0 (resp. $G_Q^\Omega(\cdot, x_0)$) of the equation $Q'(u) = 0$ in $\Omega \setminus \{x_0\}$ which has minimal growth in a neighborhood of infinity in Ω and has a removable (resp. nonremovable) singularity at x_0 .*
- (e) *Suppose that Q has a ground state φ_0 . Then there exists a positive continuous function W in Ω , such that for every $\psi \in C_0^\infty(\Omega)$ satisfying $\int_\Omega \psi \varphi_0 dx \neq 0$ there exists a constant $C > 0$ such that the following Poincaré type inequality holds:*

$$Q(u) + C \left| \int_\Omega \psi u dx \right|^p \geq C^{-1} \int_\Omega W |u|^p dx \quad \forall u \in C_0^\infty(\Omega). \quad (13.6)$$

Remarks 13.7. 1. Theorem 13.6 extends [65, Theorem 1.5] that deals with the linear case $p = 2$. The proof of Theorem 13.6 relies on the (*generalized*) *Picone identity* [5, 6].

2. We call $G_Q^\Omega(\cdot, x_0)$ (after an appropriate normalization) *the positive minimal p -Green function of the functional Q in Ω with a pole at x_0 .*

3. Suppose that $p = 2$, and that there exists a function $\psi \in L^2(\Omega)$ and $C \in \mathbb{R}$ such that

$$Q(u) + C \left| \int_\Omega \psi u dx \right|^2 \geq 0 \quad \forall u \in C_0^\infty(\Omega), \quad (13.7)$$

then the negative L^2 -spectrum of Q' is either empty or consists of a single simple eigenvalue.

We state now several positivity properties of the functional Q in parallel to the criticality theory presented in sections 2–4. For $V \in L^\infty_{\text{loc}}(\Omega)$, we use the notation

$$Q_V(u) := \int_{\Omega} (|\nabla u|^p + V|u|^p) \, dx \quad (13.8)$$

to emphasize the dependence of Q on the potential V .

Proposition 13.8. *Let $V_j \in L^\infty_{\text{loc}}(\Omega)$, $j = 1, 2$. If $V_2 \not\geq V_1$ and $Q_{V_1} \geq 0$ in Ω , then Q_{V_2} is subcritical in Ω .*

Proposition 13.9. *Let $\Omega_1 \subset \Omega_2$ be domains in \mathbb{R}^d such that $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$. Let Q_V be defined on $C_0^\infty(\Omega_2)$.*

1. *If $Q_V \geq 0$ on $C_0^\infty(\Omega_2)$, then Q_V is subcritical in Ω_1 .*
2. *If Q_V is critical in Ω_1 , then Q_V is supercritical in Ω_2 .*

Proposition 13.10. *Let $V_0, V_1 \in L^\infty_{\text{loc}}(\Omega)$, $V_0 \neq V_1$. For $s \in \mathbb{R}$ we denote*

$$Q_s(u) := sQ_{V_1}(u) + (1-s)Q_{V_0}(u), \quad (13.9)$$

and suppose that $Q_{V_j} \geq 0$ on $C_0^\infty(\Omega)$ for $j = 0, 1$.

Then the functional $Q_s \geq 0$ on $C_0^\infty(\Omega)$ for all $s \in [0, 1]$. Moreover, if $V_0 \neq V_1$, then Q_s is subcritical in Ω for all $s \in (0, 1)$.

Proposition 13.11. *Let Q_V be a subcritical in Ω . Consider $V_0 \in L^\infty(\Omega)$ such that $V_0 \not\geq 0$ and $\text{supp } V_0 \Subset \Omega$. Then there exist $0 < \tau_+ < \infty$, and $-\infty \leq \tau_- < 0$ such that Q_{V+sV_0} is subcritical in Ω for $s \in (\tau_-, \tau_+)$, and $Q_{V+\tau_+V_0}$ is critical in Ω . Moreover, $\tau_- = -\infty$ if and only if $V_0 \leq 0$.*

Proposition 13.12. *Let Q_V be a critical functional in Ω , and let φ_0 be the corresponding ground state. Consider $V_0 \in L^\infty(\Omega)$ such that $\text{supp } V_0 \Subset \Omega$. Then there exists $0 < \tau_+ \leq \infty$ such that Q_{V+sV_0} is subcritical in Ω for $s \in (0, \tau_+)$ if and only if*

$$\int_{\Omega} V_0(x) \varphi_0(x)^p \, dx > 0. \quad (13.10)$$

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